On the periodic response of a harmonically excited dry friction oscillator

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Abstract

A harmonically excited dry friction oscillator is examined analytically and numerically. We search for $2\pi/\Omega$-periodic non-sticking solutions, where Ω is the excitation frequency. Using the assumption that there are only two turnarounds during each cycle, we prove that the motion is symmetric in space and time at almost all the values of Ω. We also show that the parameter domain of non-sticking symmetric solutions is smaller than it was published in earlier contributions. The analytical results are confirmed by numerical simulation. We point out that a strange beating phenomenon may cause quite large numerical errors close to resonance.
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1. Introduction

The harmonically excited linear oscillator with viscous damping is a well-known system, its solutions can be found easily using conventional methods. However, if the oscillating body slips on a rough surface, the effects of dry friction must also be taken into account. Unfortunately, even the modelling of the frictional phenomena is a difficult task [1–4], still, there are several contributions about self-excited [5–10] and forced [11–15] vibrations of dry friction systems. The physical system, which we are concerned with, is a harmonically excited, dry friction oscillator (Fig. 1).

Dry friction force resists relative motion between contacting surfaces of the mass m and the ground. If the coefficient of dry friction is small, the body slides and its velocity is zero only for the instant when it passes through zero [13]. However, at great friction coefficient, sticking may occur: the mass remains at rest for a finite time after the velocity of the oscillator reaches zero.

The equation of motion of the analysed system is the following:

$$mz'' + cz' + kz = F_0 \cos(\omega_0\tau) - \mu mg f(z'), \tag{1}$$

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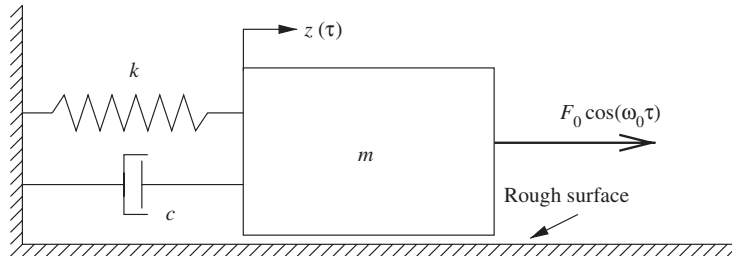*E-mail addresses:* csernak@mm.bme.hu (G. Csernák), stepan@mm.bme.hu (G. Stépán).

Fig. 1. The physical system.

where $()'$ denotes the time derivative $\mathrm{d}/\mathrm{d}\tau$,

$$f(z') = \begin{cases} 1 & \text{if } z' > 0, \\ [-\mu_1/\mu, \mu_1/\mu] & \text{if } z' = 0, \\ -1 & \text{if } z' < 0 \end{cases} \tag{2}$$

and μ, μ_1 denote the kinetic and static coefficient of friction, respectively. Thus, function $f(z')$ is meant to take (mathematically) indetermined values at zero—this corresponds to the stick phase, when the friction force adjusts itself to make equilibrium with other external forces acting on the body. To make the treatise simpler, start to measure time at one of the turnaround moments, and rescale time and displacement as $t = \tau\sqrt{k/m}$ and $x = zk/F_0$:

$$\ddot{x} + 2\alpha\dot{x} + x = \cos(\Omega(t + t_0)) - Sf(\dot{x}), \tag{3}$$

where $\dot{()}$ denotes $\mathrm{d}/\mathrm{d}t$, $\alpha = c/(2\sqrt{km})$, $\Omega = \omega_0\sqrt{m/k}$, $S = \mu mg/F_0$, and t_0 is the initial phase. Using these notations, the *condition of sticking* can be formulated as follows:

$$|x - \cos(\Omega(t + t_0))| < S_1 \quad \text{and} \quad \dot{x} = 0, \tag{4}$$

where $S_1 = \mu_1 mg/F_0 > S$.

System (3) was also studied by Deimling [16], using the theory of *differential inclusions*. He proved that in case of $\alpha > 0$ there is a single $2\pi/\Omega$-periodic solution, which is globally stable. Den Hartog [11] and Shaw [13] exploited the piecewise linear nature of the equation, and found analytically non-sticking periodic symmetric motions. Shaw introduced a method which can be used to determine the stability of periodic solutions, and showed that the $2\pi/\Omega$-periodic solutions can be unstable in case of $\alpha < 0$. In the following, we restrict ourselves to the case $\alpha = 0$. Although Shaw's contribution [13] is very profound, there are still some questions left open:

(1) Are the existing periodic solutions necessarily symmetric in time and in displacement?
(2) Does the sign of the velocity change only twice per period in case of these solutions?
(3) Does the requirement of having only two sign changes affect the stick–slip boundary?

By exploiting the piecewise linear nature of Eq. (3), explicit solutions can be found between the successive stops. The solution assumes the following form in case of positive and negative velocities, respectively:

$$x^{\pm}(t) = A^{\pm}\cos(t) + B^{\pm}\sin(t) + L\cos(\Omega t) + K\sin(\Omega t) \mp S. \tag{5}$$

The constants K and L in the particular solution can be expressed as

$$K = \frac{\sin(\Omega t_0)}{\Omega^2 - 1} \quad \text{and} \quad L = -\frac{\cos(\Omega t_0)}{\Omega^2 - 1}. \tag{6}$$

Now, if the initial conditions are given, the coefficients $A^{\pm}$ and $B^{\pm}$ can be determined. For example, if $x^-(0) = x_0$ and $\dot{x}^-(0) = 0$,

$$A^- = x_0 - S - L = x_0 - S + \frac{\cos(\Omega t_0)}{\Omega^2 - 1} \tag{7}$$

and

$$B^- = -K\Omega = -\frac{\Omega \sin(\Omega t_0)}{\Omega^2 - 1}. \tag{8}$$

Piecing together the solutions x^- and x^+ provides a general solution. Unfortunately, this matching cannot be done analytically, since the next turnaround time $t = \theta_1$ can be determined using the transcendental equation $\dot{x}^-(\theta_1) = 0$ only.

There is still possibility to find solutions analytically via searching for specific solutions. In the next section we will try to find periodic, non-sticking solutions, and try to use as few special assumptions about these solutions as possible.

2. Non-sticking symmetric solutions

Suppose that the solutions are periodic with period $T = 2\pi/\Omega = \theta_1 + \theta_2$. The periodic motion consists of two non-sticking phases that take the times θ_1 and θ_2, respectively. The velocity is negative in the first phase and positive in the second.

According to these assumptions, $\dot{x}^-(\theta_1) = 0$, and so we can obtain another expression for A^-:

$$A^- = \frac{\Omega(\sin(\Omega(t_0 + \theta_1)) - \sin(\Omega t_0)\cos(\theta_1))}{\sin(\theta_1)(\Omega^2 - 1)}. \tag{9}$$

Note, that this expression is divergent in cases $\Omega = 1$ and $\sin(\theta_1) = 0$. In the first case the well-known resonance phenomenon leads to the endless increase of the amplitude, while in the latter case an infinity of asymmetric solutions appear. The following results are valid in case of $\sin(\theta_1) \neq 0$.

If we substitute expressions (6), (8), and (9) into $x^-(t = 0)$ and $x^-(t = \theta_1)$, we can express x_0 and the displacement x_1 at the next turnaround instant:

$$x_0 = \frac{\Omega(\sin(\Omega(t_0 + \theta_1)) - \sin(\Omega t_0)\cos(\theta_1)) - \cos(\Omega t_0)\sin(\theta_1)}{\sin(\theta_1)(\Omega^2 - 1)} + S, \tag{10}$$

$$x_1 = \frac{\Omega \sin(\Omega(t_0 + \theta_1))\cos(\theta_1) - \cos(\Omega(t_0 + \theta_1))\sin(\theta_1) - \Omega \sin(\Omega t_0)}{\sin(\theta_1)(\Omega^2 - 1)} + S. \tag{11}$$

If the solution is symmetric, $x_1 = -x_0$. Thus, comparison of Eqs. (10) and (11) leads to the following condition of symmetry:

$$\sin(\theta_1)(2S(\Omega^2 - 1) - \cos(\Omega(t_0 + \theta_1)) - \cos(\Omega t_0))$$
$$+ \Omega(1 + \cos(\theta_1))(\sin(\Omega(t_0 + \theta_1)) - \sin(\Omega t_0)) = 0. \tag{12}$$

Exploiting that $x^+(\theta_1) = x_1$ and $\dot{x}^+(\theta_1) = 0$, the coefficients A^+ and B^+ can also be expressed:

$$A^+ = 2S\cos(\theta_1) + \Omega \frac{\sin(\Omega(t_0 + \theta_1)) - \sin(\Omega t_0)\cos(\theta_1)}{\sin(\theta_1)(\Omega^2 - 1)}, \tag{13}$$

$$B^+ = 2S\sin(\theta_1) - \Omega \frac{\sin(\Omega t_0)}{(\Omega^2 - 1)}. \tag{14}$$

According to our assumptions, one cycle of the periodic motion finishes at $t = \theta_1 + \theta_2$. Consequently, $x^+(\theta_1 + \theta_2) = x_0$ and $\dot{x}^+(\theta_1 + \theta_2) = 0$. Exploiting that $\theta_1 + \theta_2 = 2\pi/\Omega$ and solving these equations for the friction parameter S, one obtains two expressions, that are necessarily equal. The comparison of these expressions leads to the following equation:

$$\sin(\Omega(t_0 + \theta_1))(\sin(\theta_1) + \sin(\theta_2) - \sin(2\pi/\Omega)) = -\sin(\Omega t_0)(\sin(\theta_1) + \sin(\theta_2) - \sin(2\pi/\Omega)). \tag{15}$$

Thus, using that $\theta_1 + \theta_2 = 2\pi/\Omega$ and $\sin(\theta_1) \neq 0$, we obtain

$$\sin(\Omega(t_0 + \theta_1)) = -\sin(\Omega t_0), \tag{16}$$

which implies

$$\theta_1 = \frac{\pi}{\Omega} \quad \text{and consequently} \quad \theta_2 = \frac{\pi}{\Omega}. \tag{17}$$

Solving $\dot{x}^+(\theta_1 + \theta_2) = 0$ for $\sin(\Omega t_0)$, one obtains

$$\sin(\Omega t_0) = \frac{S(\Omega^2 - 1)\sin(\pi/\Omega)}{\Omega(\cos(\pi/\Omega) + 1)}. \tag{18}$$

Return to the examination of the condition of symmetry. Substituting $\theta_1 = \pi/\Omega$ into Eq. (12) and solving the equation for $\sin(\Omega t_0)$, the result is the same as Eq. (18). Thus, the quite trivial assumption about the period $T = 2\pi/\Omega$ implies that the non-sticking solutions are symmetric, i.e.,

$$x_1 = -x_0 \tag{19}$$

and the two phases of the cycles are of equal length.

This symmetry of solutions was usually assumed to fulfill in the literature [11,13,15], but has not been proven yet. Now, it is proven mathematically.

A simple expression can be obtained for x_0 in the following way [11,13]. Using Eq. (18) and $\dot{x}^-(\pi/\Omega) = 0$, one obtains

$$\cos(\Omega t_0) = x_0(1 - \Omega^2). \tag{20}$$

Exploiting that $\cos^2(\Omega t_0) + \sin^2(\Omega t_0) = 1$, x_0 can be expressed as

$$x_0 = \sqrt{\frac{1}{(\Omega^2 - 1)^2} - \frac{S^2 \sin^2(\pi/\Omega)}{\Omega^2(\cos(\pi/\Omega) + 1)^2}}. \tag{21}$$

This result, together with Eqs. (7) and (20) implies $A^- = -S$.

3. Validity of results

We supposed that the periodic motion consists of two non-sticking phases and the sign of the velocity is constant during each phase. To obtain non-sticking motions,

$$|x_0 - \cos(\Omega t_0)| \geqslant S_1 \tag{22}$$

must be fulfilled. Using Eq. (20), a remarkably simple slipping condition can be obtained [13]:

$$x_0 \geqslant \frac{S_1}{\Omega^2}. \tag{23}$$

For the check of the condition of having only two turnarounds per cycle, we expressed the velocity in the following form:

$$\dot{x}^-(t) = S\sin(t) - x_0\Omega\sin(\Omega t) + \frac{S\sin(\pi/\Omega)(\cos(\Omega t) - \cos(t))}{1 + \cos(\pi/\Omega)} \leqslant 0, \quad t \in [0, \pi/\Omega]. \tag{24}$$

This expression leads to

$$H(t, \Omega) := \frac{\sin(t) + \sin(\pi/\Omega)\cos(\Omega t) + \sin(t - \pi/\Omega)}{\Omega\sin(\Omega t)(1 + \cos(\pi/\Omega))} \leqslant \frac{x_0}{S}, \quad t \in [0, \pi/\Omega]. \tag{25}$$

It can be checked easily that

$$\lim_{t \to 0} H(t, \Omega) = \frac{1}{\Omega^2}, \quad \lim_{t \to 0} \dot{H}(t, 1/(2n)) = 0 \tag{26}$$

and

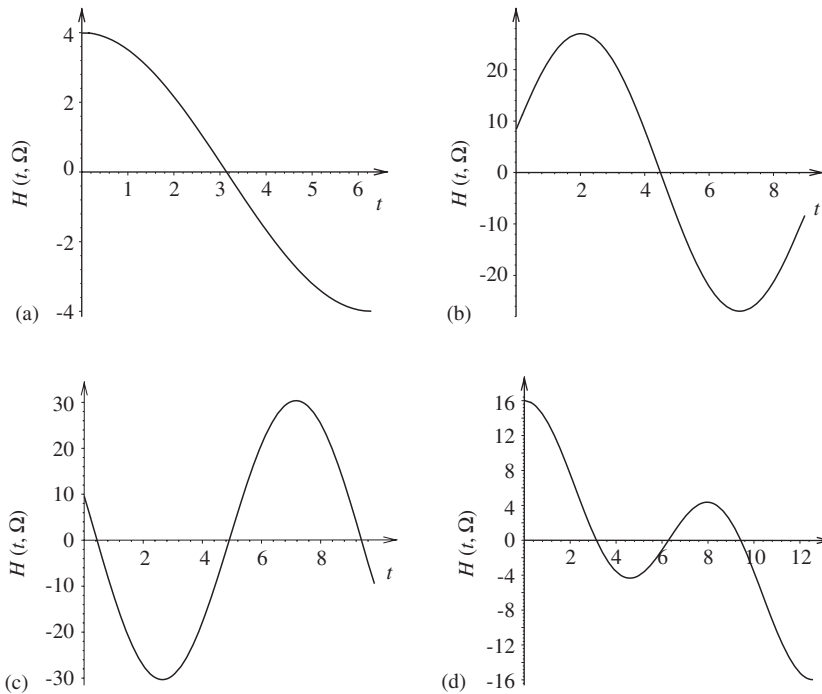$$\lim_{\Omega \to 1/(2n+1)^{\pm}} \left(\lim_{t \to 0} \dot{H}(t, \Omega) \right) = \pm\infty. \tag{27}$$

Fig. 2. Behaviour of function $H(t, \Omega)$: (a) $\Omega = 0.5$; (b) $\Omega = 0.35$; (c) $\Omega = 0.32$; (d) $\Omega = 0.25$.

Consequently, Eqs. (25) and (23) are equivalent at $t = 0$ if $S_1 = S$. Moreover, as Ω is decreased from $1/(2n)$, the maximum of H increases over $1/\Omega^2$. As Ω approaches $1/(2n + 1)$, the slope of H tends to the infinity, and then the slope changes sign.

According to these equations, and numerical evidence, we believe that the function $H(t, \Omega)$ behaves as follows. If $\Omega \geqslant \frac{1}{2}$, H takes its maximum at $t = 0$. Thus, Eq. (25) is equivalent to Eq. (23) in this domain if $S_1 = S$ (Fig. 2a). As Ω is decreased below $\frac{1}{2}$, the slope of H increases at $t = 0$. Since $H(\pi/(2\Omega), \Omega) = 0$, a local maximum appears in $t \in [0, \pi/(2\Omega)]$. This maximum is greater than $1/\Omega^2$ (Fig. 2b). At $\Omega = \frac{1}{3}$ the slope of H changes from ∞ to $-\infty$. Consequently, the point of maximum jumps from $[0, \pi/(2\Omega)]$ to $[\pi/(2\Omega), \pi/\Omega]$ (Fig. 2c). As Ω is decreased further, the local maximum decreases as well, until Ω reaches $\frac{1}{4}$, where a new maximum appears at $t = 0$ (Fig. 2d).

A similar scenario describes the behaviour of H between $\Omega = \frac{1}{4}$ and $\frac{1}{6}$, and generally, between $\Omega = 1/(2n)$ and $\Omega = 1/(2n + 2)$, too. Since some maxima of H are known to be at $t = 0$, $\Omega = 1/(2n)$, the other maxima, for $\Omega \neq 1/(2n)$ can be found quite easily using a continuation method.

According to the reasoning above, $S \max_t H(t, \Omega) = S/\Omega^2$ if $\Omega \geqslant 0.5$. Thus, the examined condition is weaker than the slipping condition (23) in this domain. Consequently, the condition of two turnarounds per period is of no importance if only sticking solutions exist in the parameter domain $\Omega \in [0, 0.5)$. Sticking motions appear exactly at $\Omega = 0.5$ if

$$x_0(\Omega = 0.5) = \frac{S_1}{0.5^2}. \tag{28}$$

The solution of Eq. (28) is $S_1 = \frac{1}{3}$. Thus, the check of condition (25) is not necessary if $S_1 > \frac{1}{3}$.

4. Numerical simulation—sticking solutions

Although non-sticking solutions can be found and analysed analytically, this does not seem to be possible in case of sticking solutions, because of the appearance of transcendental equations. Thus, to explore the possible solutions, we implemented a method that matches solutions $x^-(t)$ and $x^+(t)$ using the Newton–Raphson procedure, and uses the explicit solutions while sgn $\dot{x}$ remains constant.
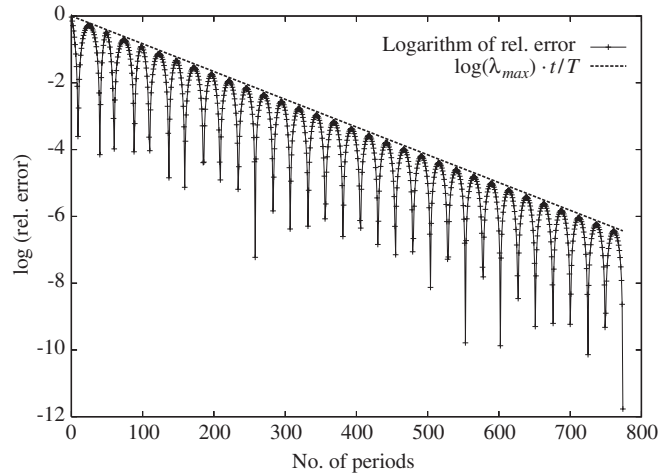
Fig. 3. Convergence to the symmetric solution at $S = 0.1$ and $\Omega = 0.98$.

Shaw [13] and Natsiavas [17] determined formulae for the stability eigenvalues $\lambda_{1,2}$ of the periodic solutions. These formulae enable us to estimate the rate of convergence of the numerical solution to Eq. (21). For the special case $S_1 = S$, there are analytically identified parameters, where linearly marginally stable asymmetric solutions appear. We tested our numerical method at these parameters, where the appropriate duration of simulation can be estimated by numerical experiments. In the following numerical examples, we performed simulations until the relative error decreased below 10^{-8}. As it is shown in Fig. 3, the decrease of the relative error of the amplitude is not uniform, but the graph has an exponential envelope curve.

Expressions (5) can be transformed to the following form:

$$
\begin{aligned}
x^{\pm}(t) &= C_1^{\pm} \sin(t + \varepsilon^{\pm}) + C_2^{\pm} \sin(\Omega t) \mp S \\
&\approx \left[2C_1^{\pm} \cos\left(\frac{1 - \Omega}{2} t + \frac{\varepsilon^{\pm}}{2} \right) + \text{const} \right] \cdot \sin\left(\frac{1 + \Omega}{2} t + \delta \right) \mp S,
\end{aligned}
\tag{29}
$$

thus, the occurrence of a beating phenomenon is apparent close to resonance. The expected period of beating is

$$
\frac{1 - \Omega}{2} T_b = \pi \implies T_b = \frac{2\pi}{|1 - \Omega|},
\tag{30}
$$

according to the textbook formula. The measured period of beating is

$$
T_m = \frac{T_b}{2} = \frac{\pi}{|1 - \Omega|}.
\tag{31}
$$

The explanation of this result is the following. The phase of the vibration suddenly changes at every switch between x^- and x^+:

$$
\Delta \varepsilon = \varepsilon^- - \varepsilon^+ = \arctan\left(\frac{A^-}{B^-} \right) - \arctan\left(\frac{A^+}{B^+} \right) = \arctan\left(\tan\left(\frac{\pi}{\Omega} \right) \right).
\tag{32}
$$

Thus, searching for the solution in $[0, \pi]$,

$$
\Delta \varepsilon =
\begin{cases}
\dfrac{\pi}{\Omega} - \pi & \text{if } 0.5 < \Omega < 1, \\
\dfrac{\pi}{\Omega} & \text{if } \Omega > 1.
\end{cases}
\tag{33}
$$

The same phase jump occurs at switches between x^+ and x^-, too. Consequently, the phase shift during one period is equal to the duration of the period, modulo 2π. This is why the period of beating is the half of T_b.

Note that—due to this beating phenomenon—the appropriate simulation time cannot be determined close to resonance by measuring the difference between successive amplitudes. Certain authors [15] published figures similar to Fig. 4a. We suppose that the visible difference between their numerical and analytical results is a consequence of short simulation times.

In Fig. 5, the numerically determined frequency–amplitude curves can be seen at four different values of the friction parameter $S = S_1$. The number N of sticks per excitation period is also indicated in the figure. The calculated amplitude curves (21) exactly coincide with the numerical results in the non-sticking cases. At low
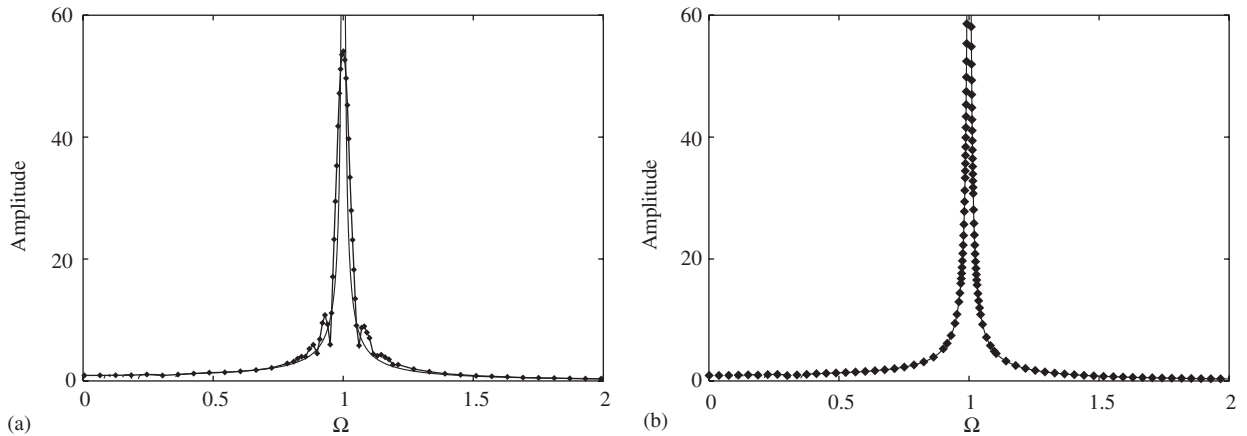

Fig. 4. Results of short and long simulations, respectively, at $S = 0.1$: (a) after 20 periods; (b) after 6000 periods. $-\blacklozenge-$ Numerical results; —— analytical results.
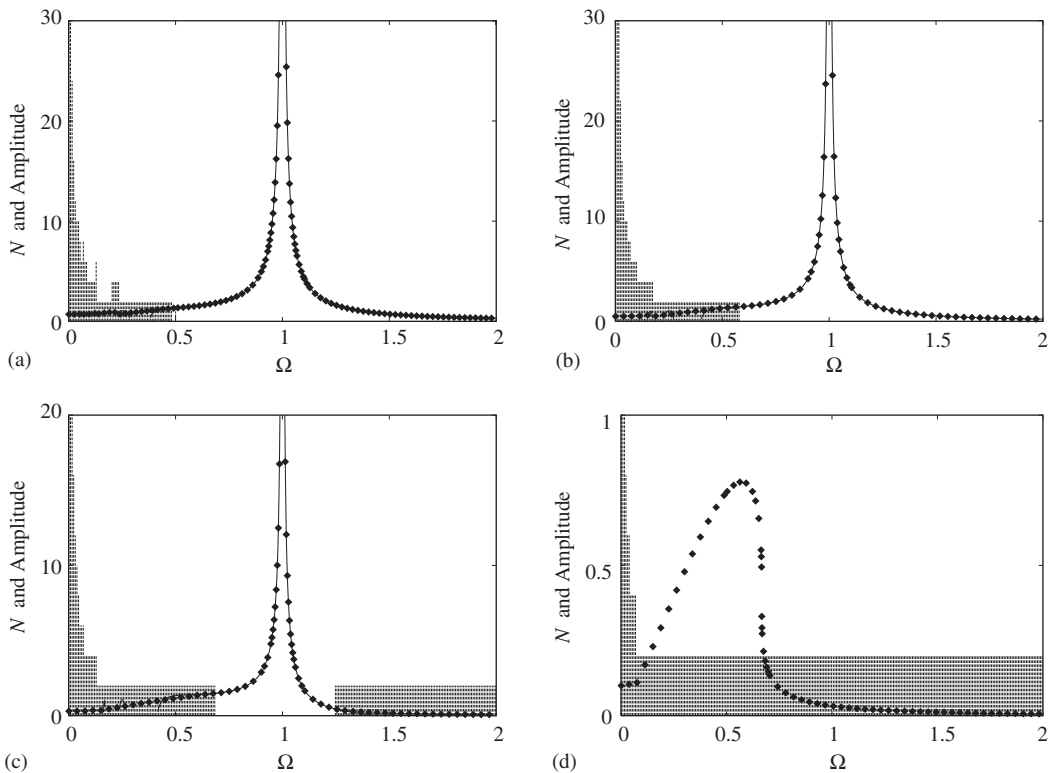

Fig. 5. Frequency-response curves at some values of the friction parameter: (a) $S = 0.3$; (b) $S = 0.5$; (c) $S = 0.7$; (d) $S = 0.9$. $\blacklozenge$ Numerically determined amplitude, —— analytical result (Eq. (21)), ▥ number of sticks (divided by 10 in case (d)).

values of S, sticking solutions appear only in the domain $\Omega \in [0, 1)$. The number of sticks per excitation period tends to infinity as Ω is decreased, as it was already shown in Ref. [12]. However, the $N(\Omega)$ diagram is not monotone decreasing at very low values of the friction parameter (see Fig. 5a). As S is increased, the local maxima of the $N(\Omega)$ diagram disappear (Fig. 5b). The further increase of S leads to the appearance of another sticking domain in $\Omega \in (1, \infty)$ (Fig. 5c). Finally, these two domains of sticking solutions merge and the amplitude assumes finite values at all values of Ω (Fig. 5d).

The critical friction coefficient, where sticking solutions appear in $\Omega \in (1, \infty)$, can be estimated as follows. At large values of Ω, we approximate $\sin(\pi/\Omega)$ by π/Ω and $\cos(\pi/\Omega)$ by 1 in Eq. (21). Thus, Eq. (23) assumes the form

$$\lim_{\Omega \to \infty} x_0(\Omega) = \lim_{\Omega \to \infty} \sqrt{\frac{1}{(\Omega^2 - 1)^2} - \frac{S^2 \pi^2}{4\Omega^2}} < \frac{S_1}{\Omega^2}. \tag{34}$$

This inequality leads to

$$S > \left(\frac{S_1}{S} + \frac{\pi^2}{4} \right)^{-1/2}. \tag{35}$$

This result has been checked numerically for the case $S_1 = S$, using a continuation method [18].

To determine the parameters, where non-sticking solutions disappear, we numerically calculated the solution of the following equations:

$$x_0(\Omega, S) - \frac{S_1}{\Omega^2} = 0,$$
$$\frac{\partial}{\partial \Omega} \left(x_0(\Omega, S) - \frac{S_1}{\Omega^2} \right) = 0. \tag{36}$$

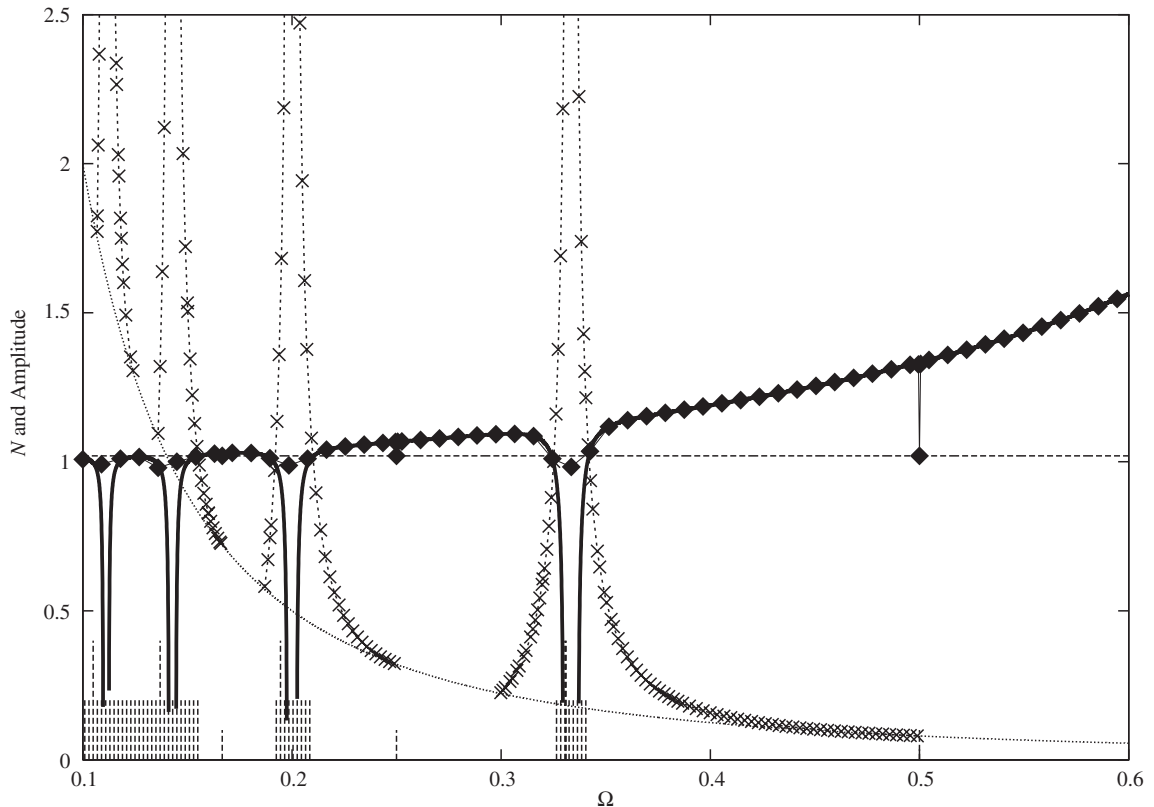For example, the root of Eq. (36) is $\Omega \approx 0.8492$, and $S \approx 0.8264$ in case $S_1 = S$.

Fig. 6. Amplitude–frequency diagram at $S = S_1 = 0.02$. $-\blacklozenge-$ Numerically determined amplitude; —— analytically determined amplitude (Eq. (21)); ⫿ number of sticks per period/10; .... stick–slip boundary I: S_1/Ω^2; $\cdots \times \cdots$ stick–slip boundary II: $S \max_t H$; - - - - $1 + S_1$.
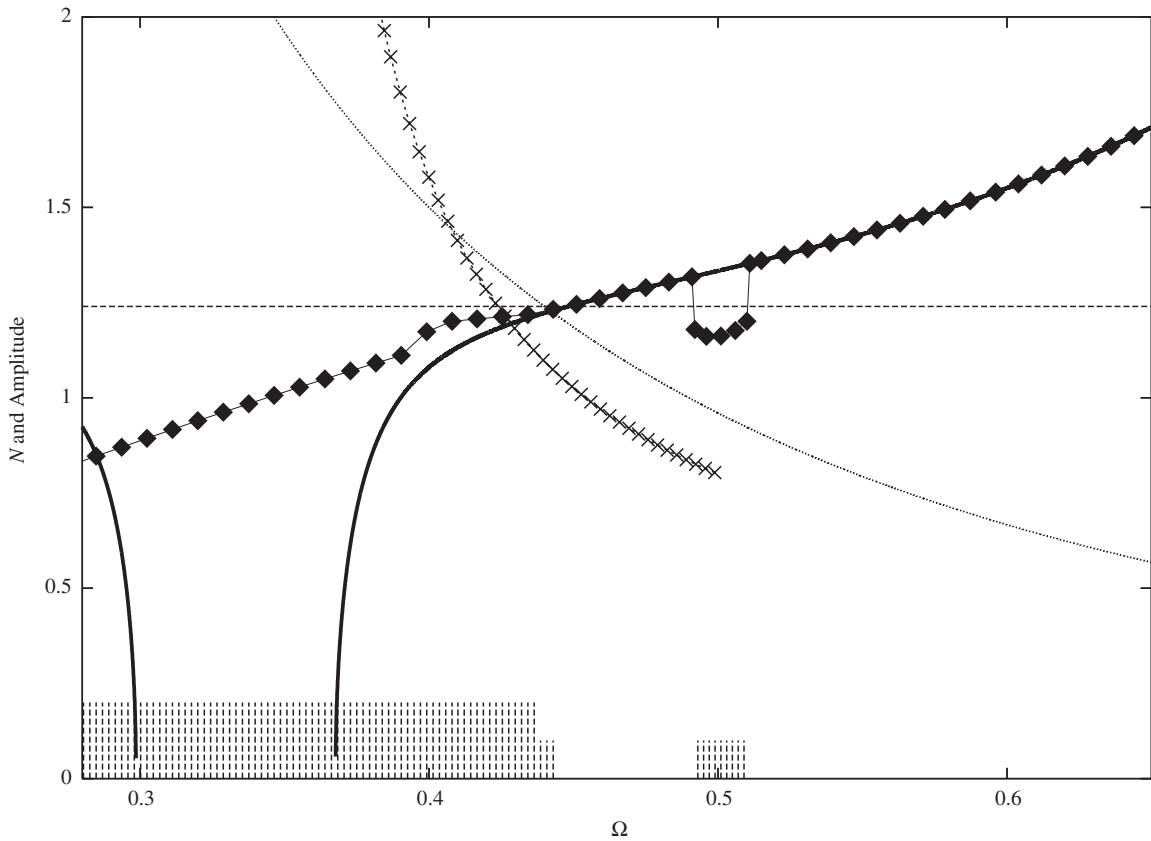
Fig. 7. Amplitude–frequency diagram at $S = 0.2$ and $S_1 = 0.24$. $- \blacklozenge -$ Numerically determined amplitude; —— analytically determined amplitude (Eq. (21)); ⬚ number of sticks per period/10; .... stick–slip boundary I: S_1/Ω^2; $\cdots \times \cdots$ stick–slip boundary II: $S \max_t H$; - - - - $1 + S_1$.

To check the validity of Eqs. (21), (23), and (25), we examined the frequency–amplitude diagram at low values of the friction parameter. In Fig. 6, the amplitude of the symmetric solution (21), the stick–slip boundary (23), and the numerically estimated $S \max_t H(t, \Omega)$ curves are shown at $S = S_1 = 0.02$, together with the numerically determined amplitude and the number of sticks per excitation period.

As it can be seen in the figure, the numerically and analytically determined amplitude curves exactly coincide in case of slipping motion, and the separation of these curves occurs according to Eqs. (23) and (25). The significance of Eq. (25) is clearly seen close to the values $\Omega = \frac{1}{3}, \frac{1}{5}$, etc., where the analytically calculated amplitude curves tend to zero, while $S \max_t H$ tends to infinity.

Numerical simulation shows that the violation of condition (25) implies the appearance of sticking behaviour. Thus, we state the conjecture that non-sticking motions with more than two stops per period either do not exist at all, or if such solutions exist, they are unstable.

Note, that for $S_1 = S$ there is a sharp peak at $\Omega = 0.5$, and there is another peak at $\Omega = 0.25$. Only one stick occurs at these parameters per period, which means that the steady-state solutions are asymmetric. As Fig. 7 shows, these solutions open up if $S_1 > S$.

5. Summary

The possible $2\pi/\Omega$-periodic, non-sticking solutions of a harmonically excited dry friction oscillator were examined. Assuming that there are only two turnarounds per period, and each period consists of two phases that take the times θ_1 and θ_2, with $\sin(\theta_1) \neq 0$, we proved, that the durations of the two phases of

the motion are equal

$$\theta_1 = \theta_2 = \frac{\pi}{\Omega} \tag{37}$$

and the greatest positive and negative amplitudes are also equal in magnitude:

$$x(0) = x(2\pi/\Omega) = x_0 = -x_1 = x(\theta_1). \tag{38}$$

The validity of the solution is limited by the above-mentioned assumptions. Although, the condition of slipping (23) has already been examined in the literature [13], the condition of two stops per period has been missing. We checked this latter condition (25), and found, that the two conditions are equivalent if $\Omega \geqslant 0.5$ and $S_1 = S$. Our numerical results showed that sticking solutions occur if either Eq. (23) or (25) are not fulfilled. Thus, we have the following conjecture: the condition of having non-sticking solutions is approximately

$$x_0 > 1 + S_1 \tag{39}$$

in case of small friction parameters (see Figs. 6 and 7), while Eq. (23) can be used if $S_1 > \frac{1}{3}$.

Acknowledgements

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